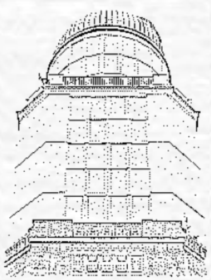
Relativistic scaling of astronomical quantities and the system of astronomical units

–

Sergei A.Klioner

Lohrmann Observatory, Dresden Technical University



Lohrmann Observatory, 4 May 2007

Content

- Relativistic time scales and reasons for them:

  TCB, TCG, proper times, TT, TDB, T_{eph}
- Scaled-BCRS
- Scaled-GCRS
- Astronomical units in Newtonian and relativistic frameworks
- Do we need astronomical units?
- TCB/TCG-, TT- and TDB-compatible planetary masses

Relativistic Time Scales: TCB and TCG

- $t = TCB$ Barycentric Coordinate Time = coordinate time of the BCRS

- $T = TCG$ Geocentric Coordinate Time = coordinate time of the GCRS

These are part of 4-dimensional coordinate systems so that

the TCB-TCG transformations are 4-dimensional: $\left(r_E^i = x^i - x_E^i(t) \right)$

$$T = t - \frac{1}{c^2}\left(A(t) + \underline{v_E^i r_E^i} \right) + \frac{1}{c^4}\left(B(t) + \underline{B^i(t) r_E^i + B^{ij}(t) r_E^i r_E^j + C\left(t, \mathbf{x}\right)} \right) + O\left(c^{-5}\right)$$

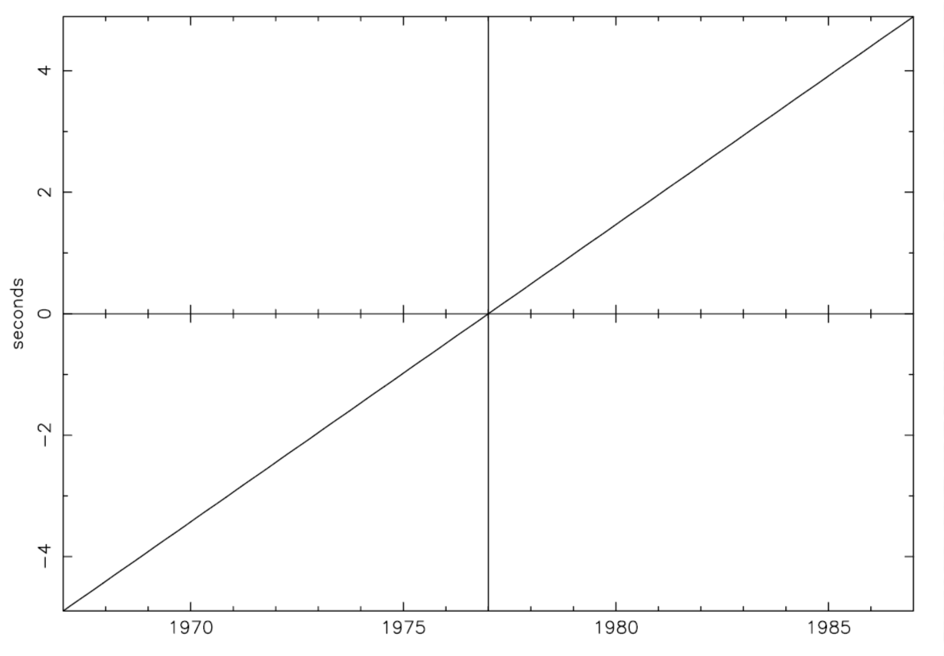
- Therefore: $TCG = TCG(TCB, \underline{x^i})$

- Only if space-time position is fixed in the BCRS $x^i = x^i_{obs}(t)$
  TCG becomes a function of TCB:

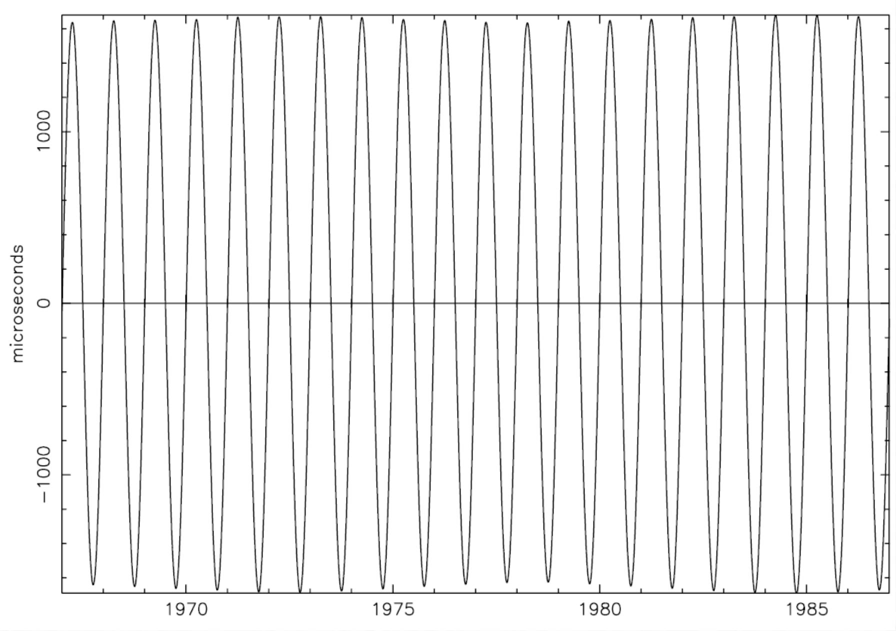
$$TCG = TCG(TCB, x^i_{obs}(TCB)) = TCG(TCB)$$

Relativistic Time Scales: TCB and TCG

- Important special case $x^i = x^i_E(t)$ gives the TCG-TCB relation at the geocenter:



linear drift removed:



Relativistic Time Scales: proper time scales

- τ proper time of each observer: what an ideal clock moving with the observer measures…

- Proper time can be related to either TCB or TCG (or both) provided that the trajectory of the observer is given:

$$x^i_{obs}(t) \quad \text{and/or} \quad X^a_{obs}(T)$$

The formulas are provided by the relativity theory:

$$\frac{d\tau}{dt} = \left(-g_{00}\left(t, \mathbf{x}_{obs}(t)\right) - \frac{2}{c} g_{0i}\left(t, \mathbf{x}_{obs}(t)\right) \dot{x}^i_{obs}(t) - \frac{1}{c^2} g_{ij}\left(t, \mathbf{x}_{obs}(t)\right) \dot{x}^i_{obs}(t)\, \dot{x}^j_{obs}(t) \right)^{1/2}$$

$$\frac{d\tau}{dT} = \left(-G_{00}\left(T, \mathbf{X}_{obs}(T)\right) - \frac{2}{c} G_{0a}\left(T, \mathbf{X}_{obs}(T)\right) \dot{X}^a_{obs}(T) - \frac{1}{c^2} G_{ab}\left(T, \mathbf{X}_{obs}(T)\right) \dot{X}^a_{obs}(T)\, \dot{X}^b_{obs}(T) \right)^{1/2}$$

Relativistic Time Scales: proper time scales

- Specially interesting case: an observer close to the Earth surface:

$$\frac{d\tau}{dT} = 1 - \frac{1}{c^2}\left(\frac{1}{2}\dot{X}^2_{obs}(T) + W_E\left(T, \mathbf{X}_{obs}\right) + \text{"tidal terms"}\right) + O\left(c^{-4}\right)$$

$\sim 10^{-17}$

- Idea: let us define a time scale linearly related to T=TCG, but which is numerically close to the proper time of an observer on the geoid:

$$TT = (1 - L_G)\, TCG, \quad L_G \equiv 6.969290134 \times 10^{-10}$$

$$\frac{d\tau}{d\,TT} = 1 - \frac{1}{c^2}\left(\text{"terms } \sim h, v^i\text{"} + \text{"tidal terms"} + \ldots\right) + \ldots$$

can be neglected in many cases

h is the height above the geoid

v^i is the velocity relative to the rotating geoid

Relativistic Time Scales: TT

- Idea: let us define a time scale linearly related to T=TCG, but which is numerically close to the proper time of an observer on the geoid:

$$TT = (1 - L_G)\,TCG, \quad L_G \equiv 6.969290134 \times 10^{-10}$$

$$\frac{d\tau}{d\,TT} = 1 - \frac{1}{c^2}\left(\text{"terms } \sim h, v^i\text{"} + \text{"tidal terms"} + \ldots\right) + \ldots$$

can be neglected in many cases

h is the height above the geoid

v^i is the velocity relative to the rotating geoid

- To avoid errors and changes in TT implied by changes/improvements in the geoid, the IAU (2000) has made L_G to be a defined constant:

$$L_G \equiv 6.969290134 \times 10^{-10}$$

- TAI is a practical realization of TT (up to a constant shift of 32.184 s)
- Older name TDT (introduced by IAU 1976): fully equivalent to TT

Relativistic Time Scales: TDB-1

- Idea: to scale TCB in such a way that the "scaled TCB" remains close to TT

- IAU 1976: TDB is a time scale for the use for dynamical modelling of the Solar system motion which differs from TT only by periodic terms.

- This definition taken literally is flawed:

  such a TDB cannot be a linear function of TCB!

  But the relativistic dynamical model (EIH equations) used by e.g. JPL is valid only with TCB and linear functions of TCB…

Relativistic Time Scales: T_{eph}

- Since the original TDB definition has been recognized to be flawed Myles Standish (1998) introduced one more time scale T_{eph} differing from TCB only by a constant offset and a constant rate:

$$T_{eph} = R \cdot TCB + T_{eph0}$$

- The coefficients are different for different ephemerides.
- The user has NO information on those coefficients from the ephemeris.
- The coefficients could only be restored by some additional numerical procedure (Fukushima's "Time ephemeris")

- T_{eph} is de facto defined by a fixed relation to TT:

  by the Fairhead-Bretagnon formula based on VSOP-87

Relativistic Time Scales: TDB-2

The IAU Working Group on Nomenclature in Fundamental Astronomy suggested to re-define TDB to be a fixed linear function of TCB:

- TDB to be defined through a conventional relationship with TCB:

$$TDB = TCB - L_B \times \left(JD_{TCB} - T_0 \right) \times 86400 + TDB_0$$

  - T_0 = 2443144.5003725 exactly,
  - $JD_{TCB} = T_0$ for the event 1977 Jan 1.0 TAI at the geocenter and increases by 1.0 for each 86400s of TCB,
  - $L_B \equiv 1.550519768 \times 10^{-8}$,
  - $TDB_0 \equiv -6.55 \times 10^{-5}$ s.

Linear drifts between time scales

Pair	Drift per year (seconds)	Difference at J2007 (seconds)
TT-TCG	0.021993	0.65979
TDB-TCB	0.489307	14.67921
TCB-TCG	0.467313	14.01939

Scaled BCRS: not only time is scaled

- If one uses scaled version TCB – T_{eph} or TDB – one effectively uses three scaling:

  - time $$t^* = F \cdot TCB + t_0^*$$
  - spatial coordinates $$\mathbf{x}^* = F \cdot \mathbf{x}$$
  - masses (μ= GM) of each body $$\mu^* = F \cdot \mu$$

$$F = 1 - L_B$$

WHY THREE SCALINGS?

Scaled BCRS

- These three scalings together leave the dynamical equations unchanged:

  - for the motion of the solar system bodies:

    (first published in 1917!)

$$\ddot{\boldsymbol{x}}_A = -\sum_{B\neq A}\mu_B\,\frac{\boldsymbol{r}_{AB}}{|\boldsymbol{r}_{AB}|^3}$$
$$+\frac{1}{c^2}\sum_{B\neq A}\mu_B\,\frac{\boldsymbol{r}_{AB}}{|\boldsymbol{r}_{AB}|^3}\Bigg\{\sum_{C\neq B}\frac{\mu_C}{|\boldsymbol{r}_{BC}|}+4\sum_{C\neq A}\frac{\mu_C}{|\boldsymbol{r}_{AC}|}+\frac{3}{2}\frac{(\boldsymbol{r}_{AB}\cdot\dot{\boldsymbol{x}}_B)^2}{|\boldsymbol{r}_{AB}|^2}$$
$$-\frac{1}{2}\sum_{C\neq A,B}\mu_C\,\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BC}}{|\boldsymbol{r}_{BC}|^3}$$
$$-2\,\dot{\boldsymbol{x}}_B\cdot\dot{\boldsymbol{x}}_B-\dot{\boldsymbol{x}}_A\cdot\dot{\boldsymbol{x}}_A+4\,\dot{\boldsymbol{x}}_A\cdot\dot{\boldsymbol{x}}_B\Bigg\}$$
$$+\frac{1}{c^2}\sum_{B\neq A}\mu_B\,\frac{\dot{\boldsymbol{x}}_A-\dot{\boldsymbol{x}}_B}{|\boldsymbol{r}_{AB}|^3}\Big\{4\,\dot{\boldsymbol{x}}_A\cdot\boldsymbol{r}_{AB}-3\,\dot{\boldsymbol{x}}_B\cdot\boldsymbol{r}_{AB}\Big\}$$
$$-\frac{1}{c^2}\frac{7}{2}\sum_{B\neq A}\frac{\mu_B}{|\boldsymbol{r}_{AB}|}\sum_{C\neq A,B}\mu_C\,\frac{\boldsymbol{r}_{BC}}{|\boldsymbol{r}_{BC}|^3}+\mathcal{O}(c^{-4}),$$

  - for light propagation:

$$c\,(t_2-t_1)=|\boldsymbol{x}_2-\boldsymbol{x}_1|$$
$$+\sum_A\frac{2\,\mu_A}{c^2}\,\ln\frac{|\boldsymbol{r}_{1A}|+|\boldsymbol{r}_{2A}|+|\boldsymbol{r}_{21}|}{|\boldsymbol{r}_{2A}|+|\boldsymbol{r}_{1A}|-|\boldsymbol{r}_{21}|}+\mathcal{O}(c^{-4}),$$

Scaled BCRS

- These three scalings lead to the following:

semi-major axes	$a^* = F \cdot a$
period	$P^* = F \cdot P$
mean motion	$n^* = F^{-1} \cdot n$
the 3rd Kepler's law	$a^3 n^2 = \mu \quad \Rightarrow \quad a^{*3} n^{*2} = \mu^*$

Scaled GCRS

- If one uses TT being a scaled version TCG one effectively uses three scaling:

  - time $T^{**} = TT = L \cdot TCG$
  - spatial coordinates $X^{**} = L \cdot X$
  - masses (μ= GM) of each body $\mu^{**} = L \cdot \mu$

$$L = 1 - L_G$$

- International Terrestrial Reference Frame (ITRF) uses such scaled GCRS coordinates and quantities

Quantities: numerical values and units of measurements

- Arbitrary quantity A can be expressed by a numerical value $\left\{ A \right\}_{XX}$ in some given units of measurements $\left[A \right]_{XX}$:

$$A = \left\{ A \right\}_{XX} \left[A \right]_{XX}$$

- XX denote a name of unit or of a system of units, like SI

- Notations taken from ISO 31-0
  "Quantities and units – Part 0: General principles", 1992

Quantities: numerical values and units of measurements

- Consider two quantities A and B, and a relation between them:

$$B = F \cdot A, \quad F = \text{const}$$

- No units are involved in this formula!

- The formula $A = \left\{A\right\}_{XX} \left[A\right]_{XX}$ should be used on both sides before numerical values can be discussed.

- In particular, $$\left\{B\right\}_{XX} = F \cdot \left\{A\right\}_{XX}$$

  is valid if and only if $$\left[B\right]_{XX} = \left[A\right]_{XX}$$

Scaled BCRS

- For the scaled BCRS this gives:

$$\{t^*\}_{XX} = F\,\{t\}_{XX},$$
$$\{\boldsymbol{x}^*\}_{XX} = F\,\{\boldsymbol{x}\}_{XX},$$
$$\{\mu^*\}_{XX} = F\,\{\mu\}_{XX},$$
$$\{a^*\}_{XX} = F\,\{a\}_{XX},$$
$$\{P^*\}_{XX} = F\,\{P\}_{XX},$$
$$\{n^*\}_{XX} = F^{-1}\,\{n\}_{XX},$$
$$\{a\}_{XX}^3\,\{n\}_{XX}^2 = \{\mu\}_{XX},$$
$$\{a^*\}_{XX}^3\,\{n^*\}_{XX}^2 = \{\mu^*\}_{XX},$$

- Numerical values are scaled in the same way as quantities if and only if the same units of measurements are used.

Units of measurements: SI

- SI units
  - time $\left[t\right]_{SI} \equiv \text{second} \equiv \text{s}$
  - length $\left[x\right]_{SI} \equiv \text{meter} \equiv \text{m}$
  - mass $\left[M\right]_{SI} \equiv \text{kilogram} \equiv \text{kg}$
- Official definitions from the SI document, published by BIPM:

The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.

The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.

Astronomical units in the Newtonian framework

- System of astronomical units
  - time $\left[t \right]_A \equiv \mathrm{day}$
  - length $\left[x \right]_A \equiv AU$
  - mass $\left[M \right]_A \equiv \mathrm{Solar\ mass} \equiv SM$

- Day is by definition 86400 SI seconds

- The mass of the Sun expressed in astronomical units of mass is by definition 1

Any possible variability of the solar mass is ignored!

Astronomical units in the Newtonian framework

- Astronomical units vs SI ones:
  - time $\left[t\right]_A = d \cdot \left[t\right]_{SI}\ , \quad d = 86400$
  - length $\left[x\right]_A = \chi \cdot \left[x\right]_{SI}$
  - mass $\left[M\right]_A = \alpha \cdot \left[M\right]_{SI}$

- AU is the unit of length with which the gravitational constant G takes the value (IAU 1938, VIth General Assembly in Stockholm)

$$\left\{G\right\}_A \equiv k^2 \equiv 0.01720209895^2$$

- AU is the semi-major axis of the [hypothetic] orbit of a massless particle which has exactly a period of $2\pi / k \approx 365.256898326328\ldots \text{days}$ in the framework of unperturbed Newtonian Keplerian motion around the Sun

Astronomical units vs. geometrized units

- Geometrized units are defined implicitly by setting three fundamental constant to unity:
  - Gravitational constant $\left\{G\right\}_{geo} = 1$
  - speed of light $\left\{c\right\}_{geo} = 1$
  - Planck constant $\left\{h\right\}_{geo} = 1$

- Why possible?

$$\left[G\right]_{SI} = m^3 kg^{-1} s^{-2}$$

$$\left[c\right]_{SI} = m\, s^{-1}$$

$$\left[h\right]_{SI} = m^2 kg\, s^{-1}$$

Astronomical units vs. geometrized units

- Both system of units – geometrized and astronomical - can be used without any relation to the “directly measurable” SI units

- The relations to the SI units of length, time and mass can only be obtained from experimental data.

Astronomical units in the Newtonian framework

- Values in SI and astronomical units:
  - distance (e.g. semi-major axis) $\{x\}_A = \chi^{-1}\{x\}_{SI}$
  - time (e.g. period) $\{t\}_A = d^{-1}\cdot\{t\}_{SI}$
  - GM $\{\mu\}_A = d^2\chi^{-3}\cdot\{\mu\}_{SI}$

- Mass parameter of a body in SI can be directly expressed as

$$\{\mu\}_{SI} = k^2 d^{-2}\chi^3 \cdot \frac{\{M\}_A}{\{M_{\text{Sun}}\}_A}$$

Astronomical units in the relativistic framework

Be ready for a mess!

Astronomical units in the relativistic framework

- Let us interpret all formulas above as TCB-compatible astronomical units

- Now let us define a different TDB-compatible astronomical units

$$\left[t\right]_{A*} \equiv \mathrm{day}^{*} = d^{*} \cdot \left[t\right]_{SI}$$

$$\left[x\right]_{A*} \equiv AU^{*} = \chi^{*} \cdot \left[x\right]_{SI}$$

$$\left[M\right]_{A*} \equiv \mathrm{SM}^{*} = \alpha^{*} \cdot \left[M\right]_{SI}$$

$$\left\{G\right\}_{A*} \equiv \left(k^{*}\right)^{2}$$

- Four constants define the system of units: $d^{*}, \chi^{*}, \alpha^{*}, k^{*}$

Astronomical units in the relativistic framework

- Considering that

$$\left\{\mu^*\right\}_{A*} = \left(k^*\right)^2 \left\{M^*_{\mathrm{Sun}}\right\}_{A*},$$

$$\left\{\mu\right\}_{A} = \left(k\right)^2 \left\{M_{\mathrm{Sun}}\right\}_{A},$$

$$\left\{\mu^*\right\}_{A*} = F \cdot \left(\frac{d^*}{d}\right)^2 \cdot \left(\frac{\chi^*}{\chi}\right)^{-3} \cdot \left\{\mu\right\}_{A}$$

- the only constraint on the constants reads

$$\frac{\left(k^*\right)^2 \left\{M^*_{\mathrm{Sun}}\right\}_{A*}}{k^2 \left\{M_{\mathrm{Sun}}\right\}_{A}} \left(\frac{\chi^*}{\chi}\right)^3 \left(\frac{d^*}{d}\right)^{-2} = F = 1 - L_B$$

Astronomical units in the relativistic framework

- Possibility I: Standish, 1995

$$d = d^* = 86400$$

$$k = k^*$$

$$\left\{ M^*_{\rm Sun} \right\}_{A*} = \left\{ M_{\rm Sun} \right\}_A = 1$$

$$\chi^* = F^{1/3} \chi$$

- This leads to

$$\left\{ x^* \right\}_{A*} = F^{2/3} \cdot \left\{ x \right\}_A$$

$$\left\{ t^* \right\}_{A*} = F \cdot \left\{ t \right\}_A$$

$$\left\{ \mu^* \right\}_{A*} = \left\{ \mu \right\}_A$$

strange scaling…

Astronomical units in the relativistic framework

- Possibility II: Brumberg & Simon, 2004; Standish, 2005

$$d = d^* = 86400$$

$$\left(k^*\right)^2 \left\{M^*_{\mathrm{Sun}}\right\}_{A*} = F \cdot k^2 \left\{M_{\mathrm{Sun}}\right\}_A$$

$$\chi^* = \chi$$

Either k is different or the mass of the Sun is not 1 or both!

- This leads to

$$\left\{x^*\right\}_{A*} = F \cdot \left\{x\right\}_A$$

$$\left\{t^*\right\}_{A*} = F \cdot \left\{t\right\}_A$$

$$\left\{\mu^*\right\}_{A*} = F \cdot \left\{\mu\right\}_A$$

The same scaling as with SI:

$$\left\{x^*\right\}_{SI} = F \cdot \left\{x\right\}_{SI}$$

$$\left\{t^*\right\}_{SI} = F \cdot \left\{t\right\}_{SI}$$

$$\left\{\mu^*\right\}_{SI} = F \cdot \left\{\mu\right\}_{SI}$$

How to extract planetary masses from the DEs

- From the DE405 header one gets:

TDB-compatible AU: $\chi^{*} = 1.49597870691 \times 10^{11}$

- Using that (also can be found in the DE405 header!)

$$\left\{\mu^{*}_{\mathrm{Sun}}\right\}_{A*} \equiv k^{2} = 2.95912208285591 1025 \times 10^{-4}$$

one gets the TDB-compatible GM of the Sun expressed in SI units

$$\left\{\mu^{*}_{\mathrm{Sun}}\right\}_{SI} = \left\{\mu^{*}_{\mathrm{Sun}}\right\}_{A*} \left(\chi^{*}\right)^{3} 86400^{-2} = 1.32712440018 \times 10^{20}$$

- The TCB-compatible GM reads
(this value can be found in IERS Conventions 2003)

$$\left\{\mu_{\mathrm{Sun}}\right\}_{SI} = \frac{1}{1-L_{B}} \left\{\mu^{*}_{\mathrm{Sun}}\right\}_{SI} = 1.32712442076 \times 10^{20}$$

Scaled GCRS

- Again three scalings ("**" denote quantities defined in the scaled GCRS; these TT-compatible quantities):
  - time $$T^{**} = L \cdot TT$$
  - spatial coordinates $$\mathbf{X}^{**} = L \cdot \mathbf{X}$$
  - masses (μ=GM) of each body $$\mu^{**} = L \cdot \mu$$
  - the scaling is fixed $$L = 1 - L_G$$
- Note that the masses are the same in non-scaled BCRS and GCRS…
- Example: GM of the Earth from SLR (Ries et al.,1992; Ries, 2005)
  - TT-compatible $$\left\{\mu^{**}_{\text{Earth}}\right\}_{SI} = \left(398600441.5 \pm 0.4\right) \times 10^6$$
  - TCG-compatible $$\left\{\mu_{\text{Earth}}\right\}_{SI} = \frac{1}{1 - L_G}\left\{\mu^{**}_{\text{Earth}}\right\}_{SI} = \left(398600441.8 \pm 0.4\right) \times 10^6$$

TCG/TCB-, TT- and TDB-compatible planetary masses

- GM of the Earth from SLR:
  - TT-compatible $\left\{\mu^{**}_{\text{Earth}}\right\}_{SI} = \left(398600441.5 \pm 0.4\right) \times 10^6$
  - TCG/B-compatible $\left\{\mu_{\text{Earth}}\right\}_{SI} = \dfrac{1}{1-L_G}\left\{\mu^{**}_{\text{Earth}}\right\}_{SI} = \left(398600441.8 \pm 0.4\right) \times 10^6$
  - TDB-compatible $\left\{\mu^{*}_{\text{Earth}}\right\}_{SI} = \left(1-L_B\right)\left\{\mu_{\text{Earth}}\right\}_{SI} = \left(398600435.6 \pm 0.4\right) \times 10^6$

Should the SLR mass be used for ephemerides?

- GM of the Earth from DE:
  - DE403 $\left\{\mu^{*}_{\text{Earth}}\right\}_{SI} = \left(398600435.6\right) \times 10^6$
  - DE405 $\left\{\mu^{*}_{\text{Earth}}\right\}_{SI} = \left(398600432.9\right) \times 10^6$

Do we need astronomical units?

- The reason to introduce astronomical units was that the angular measurements were many orders of magnitude more accurate than distance measurements.

- Arguments against astronomical units

  - The situation has changed crucially since that time!

  - Solar mass is time-dependent just below current accuracy of ephemerides

$$\dot{M}_{Sun} / M_{Sun} \sim 10^{-13}\ \mathrm{yr}^{-1}$$

  - Complicated situations with astronomical units in relativistic framework

- Why not to define AU conventionally as fixed number of meters?

- Do you see any good reasons for astronomical units in their current form?

NO!